Resonance Frequencies Calculated Efficiently With the Frequency-Domain TLM Method

Jan Hesselbarth, *Associate Member, IEEE*, and Rüdiger Vahldieck, *Fellow, IEEE*

***Abstract*—An algorithm is presented for the calculation of eigenresonances using the frequency-domain TLM method. It is based on sparse matrices and allows the computationally efficient treatment of large structures. The numerical effort and the accuracy of the solutions are discussed using two example structures.**

***Index Terms*—Eigenvalues and eigenfunctions, frequency domain analysis, microwave resonators, transmission line matrix methods.**

I. Introduction

IN THE FIELD of microwave circuit design, the determination of resonance modes and quality factors of resonators is a standard though sometimes difficult task. Utilizing a numerical method will usually lead to an eigenvalue problem. Recently, efficient methods have been developed to solve large linear, sparse eigenvalue problems [1]. Still, the number of unknowns is limited and the numerical cost is often rather large. Therefore, the numerical method used to approximate the electromagnetic fields in the resonator should give as accurate results as possible for the smallest number of unknowns. Furthermore, the numerical cost of actually finding a resonance mode with a certain accuracy has to be analyzed carefully.

In the following, the frequency-domain transmission-line matrix (FDTLM) method [2], [3] is used for analyzing resonators in terms of resonance frequencies and Q-factors. A matrix algorithm is discussed which leads to a standard sparse eigenvalue problem. This eigenvalue problem is particularly easy to solve due to a specific feature of the FDTLM approach.

II. FDTLM Method

In the FDTLM method, a brick-shaped homogeneous part of the computational domain is represented by the symmetrical condensed node (SCN) [4]. The SCN is mathematically described by a node scattering matrix of size 12 × 12 relating a vector of incident voltage waves and a vector of reflected voltage waves. The combination of the node scattering matrices of all SCN in the computational domain in the block-diagonal matrix $[S]$ leads to

$$\left(V^{ref}\right) = [S] \cdot \left(V^{inc}\right). \tag{1}$$

Manuscript received April 20, 2002; revised October 21, 2002. The review of this letter was arranged by Associate Editor Dr. Arvind Sharma.

The authors are with the Swiss Federal Institute of Technology (ETH), 8092 Zürich, Switzerland (e-mail: jhesselb@ifh.ee.ethz.ch; vahldieck@ifh.ee.ethz.ch).

Digital Object Identifier 10.1109/LMWC.2003.811044

Hereby, $[S]$ is a square matrix of size 12 N_{node}, where N_{node} denotes the number of SCN. The connection matrix connects neighboring SCN's and boundaries as

$$\left(V^{inc}\right) = [C] \cdot \left(V^{ref}\right). \tag{2}$$

Based on (1) and (2), an efficient matrix algorithm for the eigenresonance problem is developed in the following.

It is found from (1) and (2) that

$$\{[E] - [S] \cdot [C]\} \cdot \left(V^{ref}\right) = (0) \tag{3}$$

where $[E]$ denotes the unit matrix. A nontrivial solution requires the frequency-dependent matrix in (3) to be singular. All elements of $[S]$ depend on the frequency. As the matrix is sparse but large, the search for the singularity becomes rather involved: the calculation of the smallest singular value (which should become zero) or the condition number (which should become infinite) are computationally very time consuming, and the calculation of the determinant (which should become zero) quickly leads to numerical underflow. In the following, a stable and economic way is proposed to detect the frequency of singularity.

III. Cavity Resonance Frequency Algorithm

Consider first a resonator with homogeneous dielectric, regularly meshed with cubic nodes. In this particular case, all entries of $[S]$ have the same frequency dependence, namely $\exp(-jk_m\Delta/2)$, where k_m is the wavenumber and Δ denotes the geometric dimension of the node. From

$$[S] = \exp(-jk_m\Delta/2) * \left[{}^{center}S\right] \tag{4}$$

where $\left[{}^{center}S\right]$ denotes the scattering matrix at the node center, and the multiplication denoted by the asterisk is element-wise, it is found that

$$\left[{}^{center}S\right] \cdot [C] \cdot \left(V^{ref}\right) = \lambda \cdot \left(V^{ref}\right). \tag{5}$$

Equation (5) can be solved for the eigenvalue λ, from which the resonance frequency is readily found. However, it is numerically much faster to find the eigenvalue with the smallest magnitude, $\lambda_{\min}$. To that end, (5) reads

$$\{[E] - [S(f_0)] \cdot [C]\} \cdot \left(V^{ref}\right) = \lambda_{\min} \cdot \left(V^{ref}\right) \tag{6}$$

where $[S(f_0)]$ is the scattering matrix at some guess frequency f_0. A guess can be found either from some *a priori* information, from a coarse mesh run, or simply taken zero. After solving (6) for the eigenvalue $\lambda_{\min}$, the resonance frequency is given as

$$f_{res} = f_0 + \Delta f = f_0 + \frac{\ln(1-\lambda_{\min})}{j\pi\Delta\sqrt{\varepsilon\mu}}. \quad (7)$$

The better the guess frequency, f_0, the smaller the magnitude of $\lambda_{\min}$. The fundamental and the lower resonances of a resonator correspond to the smallest eigenvalues of (6) and can be found quickly by using an iterative eigensolver. Note that each physical resonance mode is accompanied by a nonphysical solution due to the nature of the TLM approach. Following the method described in [5], spurious solutions are detected easily by calculating the fields at the center of a few nodes.

For a loss-free system, the resonance frequencies are purely real. Considering losses, however, adds an imaginary part to the frequency. The Q-factor of a low-loss resonance mode is then found from

$$Q = 0.5\,|f_{res}|\,/\,|imag\,(f_{res})|\,. \quad (8)$$

Since $\lambda_{\min}$ is small, its Taylor series expansion [from (7)] for small frequency offsets Δf gives

$$\lambda_{\min} = \tfrac{1}{2}\left(\pi\Delta\sqrt{\varepsilon\mu}\right)^2 \cdot (\Delta f)^2 - j\left(\pi\Delta\sqrt{\varepsilon\mu}\right)\cdot(\Delta f)\,. \quad (9)$$

Thus, in the neighborhood of a resonance frequency, $\mathrm{Im}(\lambda_{\min}(f))$ is a linear function of frequency, and $\mathrm{Re}(\lambda_{\min}(f))$ is a quadratic function. Thus, $\lambda_{\min}(f)$ can be approximated by a parabola in the complex plane.

It is important to note that for a practical problem, a graded mesh will be used, the resonator may contain dielectric interfaces, and materials as well as boundaries may be lossy. Then, the frequency dependence cannot be extracted anymore as it has been done in (4). Fortunately, it turns out, however, that $\mathrm{Im}(\lambda_{\min}(f))$ is still an "almost linear" function of frequency in a rather wide range in the vicinity of a resonance frequency. As a result, solving (6) for two guess frequencies, f_1, f_2, allows to find a much improved frequency, f_3, by simple linear interpolation. This feature makes the search for the singularity very efficient.

IV. Cavity Resonator

Consider first a simple, air-filled, cubic cavity resonator of size 40 mm made of brass ($\sigma = 15.7 \times 10^6$ S/m). Theoretically, this resonator has a three-fold degenerated fundamental resonance at 5300 MHz with a Q-factor of 7642. A rough mesh with 64 cubic FDTLM nodes of size 10 mm (without using symmetries) will give a six-fold singularity (three physical and three spurious solutions) corresponding to $f_{res} = 5230$ MHz (-1.3%) and $Q = 7818$ (+2.3%). In this example, the rough mesh is of course responsible for the poor accuracy.

V. Dielectric Resonator

The second example is a brick-shaped, high-permittivity dielectric resonator in a perfectly conducting, air-filled cavity. The drawing in Fig. 1 shows the resonator ($\varepsilon_{rel} = 80$) of size 30 $\times$ 25 $\times$ 5 mm^3, centered in a cavity of size 50 $\times$ 50 $\times$ 15 mm^3.

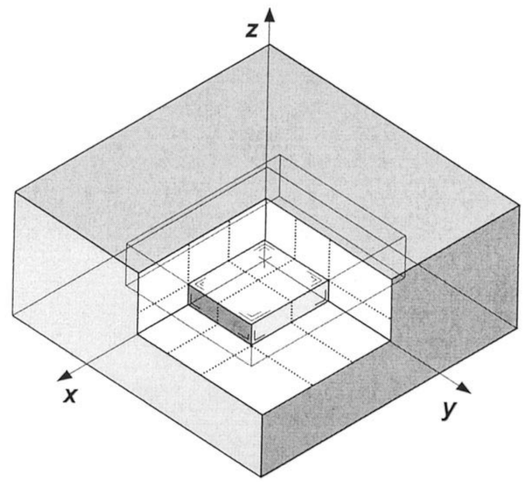

Fig. 1. Dielectric brick resonator centered in an air-filled, metallic cavity. Symmetry planes cut the computational domain by eight. The dotted lines show a rough initial mesh with 18 FDTLM node cells. Each of these cells is subsequently divided in 2^3 and 4^3 cells, respectively.

The fundamental resonating mode of the dielectric resonator resembles the TM_{101} mode of a magnetic-wall rectangular waveguide cavity. Three symmetry planes can be applied. With a finite-element solver, the resonance frequency of the fundamental mode is found at 1930.6 MHz (next higher mode at 2203.3 MHz).

In order to find the resonance frequencies using the FDTLM method, one starts with a frequency sweep on a rough mesh, which detects all singularities with low accuracy. Then, the test of spurious solutions [5] allows to cross-out nonphysical resonances. Finally, using a fine mesh, the accurate resonance frequencies are found in the vicinity of the previously detected, physical, low-accuracy singularities.

For the example resonator, the initial search on a mesh of $6 \times 6 \times 4 = 144$ nodes gives singularities near 1460 MHz and 1900 MHz. The former resonance is nonphysical, whereas the latter resonance is physical.

In a second step, the accurate resonance frequency will be found using a fine mesh ($12 \times 12 \times 8 = 1152$ nodes) in the neighborhood of 1900 MHz. The eigenvalue algorithm described above allows to find an eigenvalue $\lambda_{\min} = 0$ efficiently by linearly interpolating $\mathrm{Im}(\lambda_{\min}(f))$ over frequency. Although this is a linear function only for a cubic mesh and a homogeneous computational domain, it is still an "almost linear" function in the neighborhood of a resonance for a resonator with high dielectric contrast and noncubic node cells.

Two possible iteration paths are presented in Table I. Experience shows that a rough mesh underestimates the "true" resonance frequency. After having solved the eigenvalue problem for only three different frequencies, the linear interpolation process gives the accurate solution within reasonable limits. Note that for this mesh, the singularity at 1921.5 MHz is 0.47% off the "true" resonance.

Air-filled cavities or resonators employing low-permittivity dielectrics will show a much wider frequency band in which

TABLE I
TWO EXAMPLES FOR ITERATING $\mathrm{Im}(\lambda_{\min}(f)) = 0$ USING LINEAR INTERPOLATION. IN BOTH CASES, THE EIGENVALUE PROBLEM IS SOLVED AT 3 FREQUENCIES. THE THIRD COLUMN SHOWS THE ERROR WITH RESPECT TO THE SINGULARITY AT 1921.472 MHz

first guess f_1	f_1 = 1900 MHz	–1.12 %
second guess f_2	f_2 = 1930 MHz	+0.44 %
interpolating f_1 , f_2	f_3 = 1916.126 MHz	–0.28 %
interpolating f_2 , f_3	f_{res} = 1921.657 MHz	+0.0096 %
first guess f_1	f_1 = 1910 MHz	–0.60 %
second guess f_2	f_2 = 1920 MHz	–0.08 %
interpolating f_1 , f_2	f_3 = 1923.145 MHz	+0.09 %
interpolating f_2 , f_3	f_{res} = 1921.476 MHz	+0.0002 %

$\mathrm{Im}(\lambda_{\min}(f))$ is an "almost linear function," such that the eigenvalue problem might have to be solved only twice.

VI. CONCLUSIONS

An algorithm for the calculation of eigenresonances (frequencies, Q-factors) using the FDTLM method has been described. The problem has been cast as a sparse standard eigenvalue problem. This allows to analyze real-world, large structures by using modern eigenvalue solvers. An interesting feature of the FDTLM method has been described which gives a significant speed-up of the iterative solving process. A cavity resonator and a dielectric resonator were discussed as example structures.

REFERENCES

[1] Z. Bai, J. Demmel, J. Dongarra, A. Ruhe, and H. van der Vorst, *Templates for the Solution of Algebraic Eigenvalue Problems*. Philadelphia, PA: SIAM, 2000.

[2] H. Jin and R. Vahldieck, "The frequency-domain transmission line matrix method—A new concept," *IEEE Trans. Microwave Theory Tech.*, vol. 40, pp. 2207–2218, 1992.

[3] D. P. Johns, A. J. Wlodarczyk, A. Mallik, and C. Christopoulos, "New TLM technique for steady-state field solutions in three dimensions," *Electron. Lett.*, vol. 28, pp. 1692–1694, 1992.

[4] P. B. Johns, "A symmetrical condensed node for the TLM method," *IEEE Trans. Microwave Theory Tech.*, vol. 35, pp. 370–377, 1987.

[5] J. Hesselbarth and R. Vahldieck, "Accuracy of the frequency-domain TLM method and its application to microwave circuits," *Int. J. Numer. Modeling: Electron. Netw., Dev., and Fields*, vol. 15, pp. 371–383, 2002.